Appendix. Derivatives, Maximization, and Variance

- Here are a few derivatives that are likely to be seen in this course:

  - Derivative of a power function

    $$(x^a)' = ax^{a-1},$$

    where a is a number

  - Chain rule

    $$(f[g(x)])' = f'[g(x)]g'(x)$$

    Example:

    $$f(x) = \sqrt{x}, \quad g(x) = 1 + x^2 \quad \rightarrow \quad (\sqrt{1+x^2})' = \frac{1}{2\sqrt{1+x^2}}\, 2x = \frac{x}{\sqrt{1+x^2}}$$

  - Derivative of a ratio of two functions

    $$\left(\frac{y(x)}{g(x)}\right)' = \frac{y'(x)g(x) - g'(x)y(x)}{[g(x)]^2}$$

    Example:

    $$\left(\frac{3+x+x^2}{\sqrt{1+x^2}}\right)' = \frac{(3+x+x^2)'\sqrt{1+x^2} - (\sqrt{1+x^2})'(3+x+x^2)}{1+x^2} =$$

    $$\frac{(1+2x)\sqrt{1+x^2} - \frac{x}{\sqrt{1+x^2}}(3+x+x^2)}{1+x^2} = \frac{(1+2x)(1+x^2) - x(3+x+x^2)}{(1+x^2)^{3/2}}$$

Practice Problem

Find the derivatives of the functions

$y(x) = \sqrt{x}$

$y(x) = (1-6x^3)^{-1/2}$

$$y(x) = \frac{x^2}{(2+3x^2)}$$

- Extremum of a function is its maximum or/and minimum
- To find extremum of a function $f(x)$ we have to find its derivative, $f'(x)$, and then find the point $x^\star$ which makes the derivative equals to zero ($f'(x_0) = 0$). If the second derivative of f at x_0 is negative (positive) then $f(x)$ reaches the maximum (minimum) at x_0.

  Example. Find the extremum of the function $y(x) = 3x^2 - 4x + 6$

  $$y'(x) = (3x^2 - 4x + 6)' = 6x - 4$$

  $$y'(x_0) = 0 \quad \to \quad 6x_0 - 4 = 0 \quad \to \quad x_0 = 4/6 = 2/3$$

  Second derivative of $y(x)$ at x_0:

  $$y''(x) = (6x - 4)' = 6 > 0$$

  Therefore, we conclude that $x_0 = 2/3$ is a point at which $y(x)$ reaches its minimum of $3x_0^2 - 4x_0 + 6$

Practice Problem

Find the extremum of the following functions. Is it maximum or minimum? At what point it is reached?

$y(x) = 5 - (x - 3)^2$

$y(x) = (x+1)(30-x)+4x^2$

$y(x_1, x_2) = ax_1^2 + b(1 + x_1)x_2 + cx_2^2$, where a, b, c are constants and $x_1 + x_2 = 2$

- Now let us find variance of the portfolio that has I stocks where weight of stock i is w_i and the risky return of this stock is r_i

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\begin{aligned}
& Var(w_1r_1 + w_2r_2 + ... + w_Ir_I) = Var\left(\sum_{i=1}^{I} w_ir_i\right) = E[(w_1r_1 + w_2r_2 + ... + w_Ir_I)^2] \\
- \quad & [E(w_1r_1 + w_2r_2 + ... + w_Ir_I)]^2 = w_1^2E(r_1^2) + 2w_1w_2E(r_1r_2) + 2w_1w_3E(r_1r_3) + ... \\
+ \quad & w_2^2E(r_2^2) + 2w_2w_3E(r_2r_3) + 2w_2w_4E(r_2r_4) + ... + w_{I-1}^2E(r_{I-1}^2) + 2w_{I-1}w_IE(r_{I-1}r_I) \\
+ \quad & w_I^2E(r_I^2) - w_1^2[E(r_1)]^2 - 2w_1w_2E(r_1)E(r_2) - 2w_1w_3E(r_1)E(r_3) - ... \\
- \quad & w_2^2[E(r_2)]^2 - 2w_2w_3E(r_2)E(r_3) - 2w_2w_4E(r_2)E(r_4) - ... - w_{I-1}^2[E(r_{I-1})]^2 \\
- \quad & 2w_{I-1}w_IE(r_{I-1})E(r_I) - w_I^2[E(r_I)]^2 \\
= \quad & w_1^2(E(r_1^2) - [E(r_1)]^2) + 2w_1w_2[E(r_1r_2) - E(r_1)E(r_2)] + 2w_1w_3[E(r_1r_3) - E(r_1)E(r_3)] \\
+ \quad & w_2^2(E(r_2^2) - [E(r_2)]^2) + 2w_2w_3[E(r_2r_3) - E(r_2)E(r_3)] + 2w_2w_4[E(r_2r_4) - E(r_2)E(r_4)] \\
+ \quad & w_{I-1}^2(E(r_{I-1}^2) - [E(r_{I-1})]^2) + 2w_{I-1}w_I[E(r_{I-1}r_I) - E(r_{I-1})E(r_I)] \\
+ \quad & w_I^2(E(r_I^2) - [E(r_I)]^2) \\
= \quad & \sum_{i=1}^{I} w_i^2Var(r_i) + 2\sum_{i=1}^{I} w_1w_iCov(r_1, r_i) + 2\sum_{i=2}^{I} w_2w_iCov(r_2, r_i) + 2w_{I-1}w_ICov(r_{I-1}, r_I) \\
= \quad & \sum_{i=1}^{I} w_i^2Var(r_i) + \sum_{i=1, i\neq j}^{I} \sum_{j=1}^{I} w_iw_jCov(r_i, r_j)
\end{aligned}
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